Problem Set 1

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1. Solution

(a)

Let's start by looking at the functional derivative of the generating functional for the connected correlation functions:

$$\frac{\delta W[J]}{\delta J(x)} = i\frac{\delta}{\delta J(x)}\log(Z) = -\frac{\int \mathcal{D}\phi e^{\int(\mathcal{L}+J\phi)}\phi(x)}{\mathcal{D}\phi e^{\int(\mathcal{L}+J\phi)}}, \tag{1}$$

where Z is the generating functional of all correlation functions. Identifying the right hand side as the classical field,

$$-\frac{\int \mathcal{D}\phi e^{\int(\mathcal{L}+J\phi)}\phi(x)}{\mathcal{D}\phi e^{\int(\mathcal{L}+J\phi)}} = -\langle\Omega|\phi(x)|\Omega\rangle \equiv \phi_{cl}(x), \tag{2}$$

we can write

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)} = \frac{\delta\phi_{cl}(z)}{\delta J(x)}. \tag{3}$$

Now, let's remember that the generating functional for one particle irreducible correlation functions is the effective action: $\Gamma[\phi_{cl}] \equiv -W[J] - \int d^4y J(y)\phi_{cl}(y)$. Its functional derivative with respect to the classical action is (Peskin 11.48)

$$\frac{\delta}{\delta\phi_{cl}(x)}\Gamma[\phi_{cl}] = -\frac{\delta}{\delta\phi_{cl}(x)}W[J] - \int d^4y\frac{\delta J(y)}{\delta\phi_{cl}(x)}\phi_{cl}(y) - J(x). \tag{4}$$

Rewriting the first term using the chain rule for functional derivatives,

$$\frac{\delta}{\delta\phi_{cl}(x)}W[J] = \int d^4y\frac{\delta J(y)}{\delta\phi_{cl}(x)}\frac{\delta W[J]}{\delta J(y)}, \tag{5}$$

and again noting the fact that $\frac{\delta W[J]}{\delta J(y)} = -\phi_{cl}(y)$, we now see that the first two terms cancel:

$$\frac{\delta}{\delta\phi_{cl}(x)}\Gamma[\phi_{cl}] = +\int d^4y\frac{\delta J(y)}{\delta\phi_{cl}(x)}\phi_{cl}(y) - \int d^4y\frac{\delta J(y)}{\delta\phi_{cl}(x)}\phi_{cl}(y) - J(x) = -J(x). \tag{6}$$

Taking another derivative, we find

$$\frac{\delta^2\Gamma[\phi]}{\delta\phi(x)\delta\phi(y)} = -\frac{\delta J(y)}{\delta\phi_{cl}(x)}. \tag{7}$$

Combining these two results, choosing a prudent order of differentiation, and again making use of the chain rule for functional derivatives, we arrive at the equality

$$\int d^4z\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(y)} = -\int d^4z\frac{\delta\phi_{cl}(x)}{\delta J(z)}\frac{\delta J(z)}{\delta\phi_{cl}(y)} = -\frac{\delta\phi_{cl}(x)}{\delta\phi_{cl}(y)} = -\delta^{(4)}(x-y) \tag{8}$$

$\frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}$ is the connected two-point function, aka, the propagator of ϕ. Thus we identify $\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\delta\phi(y)}$ as the negative of the inverse propagator!

(b)

For higher derivatives, it is helpful to again use the functional derivative chain rule. In particular, note that we can write

$$\frac{\delta}{\delta J(z)} = \int d^4w \frac{\delta \phi_{cl}}{\delta J(z)} \frac{\delta}{\delta \phi_{cl}(w)} = i \int d^4w D(z,w) \frac{\delta}{\delta \phi_{cl}(w)}, \tag{9}$$

where in the last equality I have used $\frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} = -iD(x,y)$.

The last ingredient we need is the relation for the differential of an inverse matrix

$$\frac{\partial}{\partial \alpha} M^{-1}(\alpha) = -M^{-1} \frac{\partial M}{\partial \alpha} M^{-1} \tag{10}$$

Above, we argued that

$$\frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} = -\Big(\frac{\delta^2 \Gamma[\phi_{cl}]}{\delta \phi_{cl}(z) \delta \phi_{cl}(y)}\Big)^{-1}. \tag{11}$$

So combining these identities, we can find the next higher derivative of W.

$$\begin{aligned} \frac{\delta W[J]}{\delta J_x \delta J_y \delta J_z} &= -i \int d^4w D(z,w) \frac{\delta}{\delta \phi_{cl}(w)} \Big(\frac{\delta^2 \Gamma[\phi_{cl}]}{\delta \phi_{cl}(x) \delta \phi_{cl}(y)}\Big)^{-1} \\ = -i \int d^4w D(z,w)(-1) \int d^4u \int d^4v (iD(x,u)) &\frac{\delta^3 \Gamma}{\delta \phi_{cl}(u) \delta \phi_{cl}(v) \delta \phi_{cl}(w)} (iD(v,y)) \\ = -i \int d^4u \int d^4v \int d^4w D(x-u) D(y-v) D(z-w) &\frac{\delta^3 \Gamma}{\delta \phi_{cl}(u) \delta \phi_{cl}(v) \delta \phi_{cl}(w)} \end{aligned} \tag{12}$$

Now, let's kill the propagators in the last line by multiplying both sides by the corresponding inverse propagators and integrating over them.

$$\frac{\delta^3 \Gamma}{\delta \phi_{cl}(x) \delta \phi_{cl}(y) \delta \phi_{cl}(z)} = \int d^4u \int d^4v \int d^4w D^{-1}(x-u) D^{-1}(y-v) D^{-1}(z-w) \frac{\delta W[J]}{\delta J_x \delta J_y \delta J_z}. \tag{13}$$

We now see that Γ generates the connected Green function with the external propagators removed!

2. Solution

(a)

(Following Peskin p. 290) The generating functional of correlation functions Z is given by

$$Z[J] \equiv \int \mathcal{D}\phi \exp\Big[i \int d^4x [\mathcal{L} + J(x)\phi(x)]\Big]. \tag{14}$$

In ϕ^4 theory, this is written as

$$\int \mathcal{D}\phi \exp\Big[i \int d^4x [\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 + J(x)\phi(x)]\Big]. \tag{15}$$

For notational convenience, I define a new variable

$$\alpha \equiv \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J(x)\phi(x) \tag{16}$$

Now expanding the exponential to first order in λ gives

$$\int \mathcal{D}\phi\Big[\exp[i\int d^4x\alpha]\Big(1-i\int d^4y\frac{\lambda}{4!}\phi^4\Big)\Big]. \tag{17}$$

Notice that taking derivatives of the form $\frac{\delta\alpha}{\delta J(x)}$ give factors of $i\phi(x)$. We can exploit this trick to remove ϕ from the path integral.

$$Z[J]=\Big(1-i\frac{\lambda}{4!}\int d^4y\frac{\delta^4}{\delta^4 J(y)}\Big)\int \mathcal{D}\phi\Big[\exp[i\int d^4x\alpha]\Big] \tag{18}$$

Now Lets look at the $\int d^4x\alpha$ term:

$$\int d^4x\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2+J(x)\phi(x) \tag{19}$$

Integrating the kinetic term by parts and inserting a convergence term, we can rewrite this as

$$\int d^4x\frac{1}{2}\phi(-\partial^2-m^2+i\epsilon)\phi+J(x)\phi(x). \tag{20}$$

Let's shift the integration (Jacobian = 1) by introducing a new field

$$\phi'(x)=\phi(x)-i\int d^4yD_F(x-y)J(y) \tag{21}$$

, where D_F is the free field propagator (and the Green's function of the Klein Gordon operator). Now, our Eq. 19 becomes

$$\int d^4x[\frac{1}{2}\phi'(-\partial^2-m^2+i\epsilon)\phi']-\int d^4x\int d^4y\frac{1}{2}J(x)[-iD_F(x-y)]J(y). \tag{22}$$

Plugging this back in to our previous expression for Z, we get

$$\Big(1-i\frac{\lambda}{4!}\int d^4y\frac{\delta^4}{\delta^4 J(y)}\Big)\int \mathcal{D}\phi'\Big[\exp[i\int d^4x\mathcal{L}_0(\phi')-i\int d^4x\int d^4y\frac{1}{2}J(x)[-iD_F(x-y)]J(y)]\Big]. \tag{23}$$

Now the terms only depend on ϕ' or J, but not both. Defining the constant

$$\mathcal{N}=\int \mathcal{D}\phi'\Big[\exp[i\int d^4x\mathcal{L}_0(\phi')]\Big], \tag{24}$$

we finally arrive at

$$Z[J]=\mathcal{N}\Big(1-i\frac{\lambda}{4!}\int d^4y\frac{\delta^4}{\delta^4 J(y)}\Big)\exp\Big[-\frac{1}{2}\int d^4x\int d^4yJ(x)[D_F(x-y)]J(y)\Big] \tag{25}$$

Now taking the 4 functional derivatives, we can write Z as

$$\begin{aligned}&\mathcal{N}\Big(\exp[-\frac{1}{2}\int d^4x\int d^4yJ(x)[D_F(x-y)]J(y)]\Big)\\ &\times\Big[1-i\frac{\lambda}{4!}\Big(\int d^4y(\int dx_1D_F(y-x_1)J(x_1))^4+6D_F(0)(\int dx_1D_F(y-x_1)J(x_1))^2+3D_F^2(0)\Big)\Big].\end{aligned} \tag{26}$$

Our normalization condition that $Z[0]=1$ (no sources), implies $\mathcal{N}(1-i\frac{\lambda}{8}\int d^4yD_F^2(0))=1$. Since $Z[J]=\exp\{iW[J]\}$, and we've only kept terms that are first order in λ, it follows that everything multiplying $\mathcal{N}$, without the exponential, is equal to W. That is,

$$\begin{aligned}W[J]=&-\frac{1}{2}\int d^4x\int d^4yJ(x)[D_F(x-y)]J(y)]\Big)\\ &-\frac{i\lambda}{4!}\Big(\int d^4y(\int dxD_F(y-x)J(x))^4+6D_F(0)(\int dxD_F(y-x)J(x))^2\Big)\end{aligned} \tag{27}$$

(b)

We'll get the 4-point connected Green's function by taking 4 functional derivatives of W with respect to J and evaluating them at J=0. Clearly only the J^4 term will contribute. Each derivative will yield a $\delta(y-x_i)$ which will set the argument of the Feynman propagator to be $(y-x_i)$. Keeping careful track of the numerical factors, we'll find

$$\langle\Omega|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle = -i\lambda\int d^4y D_F(y-x_1)D_F(y-x_2)D_F(y-x_3)D_F(y-x_4) \quad (28)$$

To get the Feynman rule in momentum space, we simply Fourier transform this expression. Writing the Feynman propagator as

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik(x-y)}, \quad (29)$$

We see that each of the

$$\int d^4x_i e^{-ik_i(x_i-y)}D_F(y-x_i)$$

terms gives

$$\int d^4x_i e^{-ik_i(x_i-y)}\int \frac{d^4k_i}{(2\pi)^4}\frac{i}{k_i^2-m^2+i\epsilon}e^{-ik_i(x_i-y)} = \frac{i}{k_i^2-m^2+i\epsilon} \quad (30)$$

after performing the integral over k to get a delta function, then using that to kill the 2nd integral.

The momentum space Feynman rule is thus

$$\lambda\frac{1}{k_1^2-m^2+i\epsilon}\frac{1}{k_2^2-m^2+i\epsilon}\frac{1}{k_3^2-m^2+i\epsilon}\frac{1}{k_4^2-m^2+i\epsilon}. \quad (31)$$

This represents four particles propagating to an interaction point with a vertex of $-i\lambda$. (somewhere a factor of $-i$ went missing, so I made a withdrawal from the sign bank.)

(c)

As discussed in problem 1, The classical field is defined by

$$\frac{\delta W[J]}{\delta J(y)} = -\phi_{cl}(y). \quad (32)$$

So to find ϕ_{cl} perturbatively, let's take a functional derivative of W from Eq. (27) with respect to J. The derivative of the first term gives two identical terms with only one factor of J after relabeling integration variables. Similarly for the J^2 term and the J^4 term, keeping in mind that there are 2 and 4 ways to take the derivative respectively. Keeping track of numerical factors, the net result is

$$\begin{aligned}\frac{\delta W[J]}{\delta J(w)} = \phi_{cl}(w) = -\int d^4y D_F(w-y)J(y) \\ -i\lambda\Big(\frac{1}{6}\int d^4y\int d^4x_i D_F(y-w)D_F(y-x_1)D_F(y-x_2)D_F(y-x_3)J(x_1)J(x_2)J(x_3) \\ +\frac{1}{2}D_F(0)\int d^4y\int d^4x D_F(y-w)D_F(y-x)J(x)\Big)\end{aligned} \quad (33)$$

Noting that D_F is the Green's function of the Klein Gordon operator,$(\Box+m^2)D_F(x-y) = -\delta^{(4)}(x-y)$, we operate on both sides.

$$(\Box + m^2)\phi_{cl}(w) = J(w) + i\lambda\Big(\frac{1}{6}\int d^4x_i D_F(w-x_1)D_F(w-x_2)D_F(w-x_3)J(x_1)J(x_2)J(x_3) + \frac{1}{2}D_F(0)\int d^4x D_F(w-x)J(x)\Big) \tag{34}$$

Now we can solve this order by order for J. To zeroth order in λ, $J(w) = (\Box + m^2)\phi_{cl}(w)$. Substituting this in to the first order terms, we have

$$\begin{aligned}(\Box + m^2)\phi_{cl}(w) = J(w) + i\lambda\Big(\frac{1}{6}\int d^4x_i D_F(w-x_1)D_F(w-x_2)D_F(w-x_3) \\ \times(\Box + m^2)\phi_{cl}(x_1)(\Box + m^2)\phi_{cl}(x_2)(\Box + m^2)\phi_{cl}(x_3) \\ + \frac{1}{2}D_F(0)\int d^4x D_F(w-x)(\Box + m^2)\phi_{cl}(x)\Big)\end{aligned} \tag{35}$$

Now integrating by parts, we can move the Klein Gordon operators to act on the Feynman propagators, giving us some delta functions.

$$(\Box + m^2)\phi_{cl}(w) = J(w) + i\lambda\Big(-\frac{1}{6}\int d^4x_i \delta^{(4)}(w-x_1)\delta^{(4)}(w-x_2)\delta^{(4)}(w-x_3)\phi_{cl}(x_1)\phi_{cl}(x_2)\phi_{cl}(x_3) - \frac{1}{2}D_F(0)\int d^4x \delta^{(4)}(w-x)\phi_{cl}(x)\Big) \tag{36}$$

Integrating over the deltas, we end up with

$$(\Box + m^2)\phi_{cl}(w) = J(w) - i\lambda\big(\frac{1}{6}\phi_{cl}(w)^3 + \frac{1}{2}D_F(0)\phi_{cl}(w)\big) \tag{37}$$

Thus, we have found J to first order in λ:

$$J(w) = (\Box + m^2)\phi_{cl}(w) + i\lambda\big(\frac{1}{6}\phi_{cl}(w)^3 + \frac{1}{2}D_F(0)\phi_{cl}(w)\big). \tag{38}$$

We can now find the classical action by substituting this in to the expression

$$\Gamma[\phi_{cl}] \equiv -W[J] - \int d^4y J(y)\phi_{cl}(y). \tag{39}$$

Recalling that the generating functional for connected diagrams, W, looks like

$$\begin{aligned}W[J] = -\frac{1}{2}\int d^4x \int d^4y J(x)[D_F(x-y)]J(y)]\Big) \\ -\frac{i\lambda}{4!}\Big(\int d^4y\big(\int dx D_F(y-x)J(x)\big)^4 + 6D_F(0)\big(\int dx D_F(y-x)J(x)\big)^2\Big)\end{aligned} \tag{40}$$

we now substitute in the first order expression for J and discard any terms that are order λ^2. We again make use of some integration by parts trickery to move the KG operator and create delta functions which kill the integrals over y.

$$\begin{aligned}W[J] = -\frac{1}{2}\int d^4x\big(\phi_{cl}(x)\big)(\Box + m^2)\phi_{cl}(w) + i\lambda\big(\frac{1}{6}\phi_{cl}(w)^3 + \frac{1}{2}D_F(0)\phi_{cl}(w)\big) \\ -\frac{i\lambda}{4!}\int d^4x \phi_{cl}(x)^4 - \int d^4x \frac{6i\lambda}{4!}D_F(0)\phi_{cl}(x)^2.\end{aligned} \tag{41}$$

The 2nd term in the RHS of (39) is simply the integral over (38) with an additional factor of $\phi_{cl}(y)$. This term then gives us

$$-\int d^4x\phi_{cl}(x)[(\Box+m^2)\phi_{cl}(x)-i\lambda(\frac{1}{6}\phi_{cl}(x)^3-\frac{1}{2}D_F(0)\phi_{cl}(x))]. \quad (42)$$

Thus, we can finally write the classical action as

$$\Gamma[\phi_{cl}]=-\frac{1}{2}\int d^4x\Big[\phi_{cl}(x)(\Box+m^2)\phi_{cl}(x)-\frac{i\lambda}{4}D_F(0)\phi_{cl}(x)^2-\frac{i\lambda}{8}\phi^4_{cl}(x)\Big]. \quad (43)$$

The Feynman rule (in position space) can now be generated by taking functional derivatives of the classical action with respect to the classical field. At this point, I note that the numerical factor on the ϕ^4_{cl} term should be $\frac{1}{4!}$. If you'll excuse a missing factor of 3, we find the 4 point interaction to be

$$-i\lambda\int d^4x\delta^{(4)}(x-x_1)\delta^{(4)}(x-x_2)\delta^{(4)}(x-x_3)\delta^{(4)}(x-x_4) \quad (44)$$

Going to momentum space, we integrate over the delta functions and pick up some factors of 2π and a momentum conservation delta for our troubles. Discarding these, the final momentum space Feynman rule is simply $(-i\lambda)$

3. Solution

Starting from the path integral definition of the generating functional,

$$Z[J]=\mathcal{N}\int\mathcal{D}\phi\exp\Big\{[i\int d^4x[\mathcal{L}+J(x)\phi(x)]\Big\}, \quad (45)$$

we perform a change of variables $\phi(x)\to\phi(x)+\epsilon(x)$, where $\epsilon(x)$ is an arbitrary infinitesimal function of x. Our generating functional is now

$$Z[J]=\mathcal{N}\int\mathcal{D}\phi\exp\Big\{[i\int d^4x[\mathcal{L}+J(x)\phi(x)\Big\}\exp\Big\{i\int d^4xJ(x)\epsilon(x)]\Big\}, \quad (46)$$

Expanding the second exponential to first order in ϵ gives us

$$\mathcal{N}\int\mathcal{D}\phi\exp\Big\{[i\int d^4x\mathcal{L}+J(x)\phi(x)\Big\}\Big(1+i\int d^4xJ(x)\epsilon(x)\Big) \quad (47)$$

Since the equation of motion of this scalar field is $\Box\phi(x)+V'(\phi)=0$, we can insert an additional term with no penalty

$$Z[J]=\mathcal{N}\int\mathcal{D}\phi\exp\Big\{[i\int d^4x\mathcal{L}+J(x)\phi(x)\Big\}\Big(1+i\int d^4x\epsilon(x)-\Box\phi-V'(\phi)+J(x)\Big) \quad (48)$$

Since the first term is simply equal to $Z[J]$ (which is unchanged by our change of variables), we have the following equality

$$\int\mathcal{D}\phi\exp\Big\{i\int d^4x[\mathcal{L}+J(x)\phi(x)]\Big\}\int d^4x\epsilon(x)\big(-\Box\phi-V'(\phi)+J(x)\big)=0 \quad (49)$$

Since $\epsilon(x)$ was arbitrary, we can now choose $\epsilon(x)=\epsilon\delta^{(4)}(x-y)$, where ϵ is now an infinitesimal constant. Plugging this in, we have

$$\int\mathcal{D}\phi\exp\Big\{i\int d^4x[\mathcal{L}+J(x)\phi(x)]\Big\}\int d^4x[\epsilon\delta^{(4)}(x-y)\big(-\Box\phi-V'(\phi)+J(x)\big)]=0. \quad (50)$$

Integrating over the delta, we have

$$\int \mathcal{D}\phi \exp\Big\{i\int d^4x[\mathcal{L}+J(x)\phi(x)]\Big\}\epsilon\big(-\Box\phi - V'(\phi)+J(y)\big)=0. \tag{51}$$

Now, taking a functional derivative of (50) with respect to $J(x)$, we find

$$\begin{aligned}\int \mathcal{D}\phi\Big[&\exp\Big\{i\int d^4x[\mathcal{L}+J(x)\phi(x)]\Big\}\phi(x)\epsilon(-\Box\phi - V'(\phi))\\ &+\epsilon\exp\Big\{i\int d^4x[\mathcal{L}+J(x)\phi(x)]\Big\}\delta^{(4)}(y-x)\Big]=0\end{aligned} \tag{52}$$

In the limit of no background fields (J=0), this simplifies to

$$\int \mathcal{D}\phi \exp\Big\{i\int d^4x\phi(x)\Big\}\Big[-(\Box\phi+V'(\phi))+\delta^{(4)}(y-x)\Big]=0 \tag{53}$$

We can now use the definition of the generating functional (noting that the normalization is unimportant as it will cancel out)

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \frac{\int\mathcal{D}\phi\phi(x)\phi(y)\exp(iS[\phi])}{\int\mathcal{D}\phi\exp(iS[\phi])}, \tag{54}$$

to express this result as

$$\Box_x\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = -\langle\Omega|T\{V'(\phi(x))\phi(y)\}|\Omega\rangle - i\delta^{(4)}(y-x). \tag{55}$$

4 Solution

(a)

The free-field Feynman propagator in coordinate space for the Klein Gordon Lagrangian is

$$D_F(x)=\langle 0|T\{\phi(x)\phi(0)\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ipx}}{p^2-m^2+i\epsilon} \tag{56}$$

where I have taken $y=0$ for notational convenience. We first integrate over p_0, using the fact that $E=(\vec{p}^{\,2}+m^2)$

$$\int dp_0\frac{e^{-ip_0t}}{p_0^2-E^2+i\epsilon}. \tag{57}$$

Performing the integration in the complex plane, we can separate the poles into two terms using partial fractions

$$\frac{1}{p_0^2-E^2+i\epsilon}=\frac{1}{p_0-(E-i\epsilon)}\frac{1}{p_0+(E-i\epsilon)}=\frac{1}{2E}\Big[\frac{1}{p_0-(E-i\epsilon)}-\frac{1}{p_0+(E-i\epsilon)}\Big] \tag{58}$$

where we have dropped terms of $\mathcal{O}(\epsilon^2)$ and relabeled $2E\epsilon=\epsilon$, noting that we will take the limit $\epsilon\to 0$ at the end of the calculation.

The integral over the first term has a pole at $p_0=E-i\epsilon$. If $t>0$, we close the contour in the upper region of the complex plane, enclosing no poles. By the residue theorem, the integral yields 0. If $t<0$, we close the contour in the lower region of the complex plane, picking up a residue and a minus sign for taking a clockwise path. The integral is then

$$\int_{-\infty}^{\infty}\frac{dp_0e^{ip_0t}}{p_0-(E-i\epsilon)}=-2\pi ie^{iEt}\theta(-t). \tag{59}$$

Similarly, the second term yields

$$\int_{-\infty}^{\infty} \frac{dp_0 e^{ip_0 t}}{p_0 + (E - i\epsilon)} = 2\pi i e^{iEt}\theta(t). \tag{60}$$

In the limit of vanishing ϵ, the total integral over p_0 then gives

$$\int dp_0 \frac{e^{-ip_0 t}}{p_0^2 - E^2 + i\epsilon} = -\frac{i\pi}{E}\left(e^{iEt}\theta(-t) + e^{-iEt}\theta(t)\right). \tag{61}$$

The Feynman propagator now looks like

$$\int \frac{d^3p}{(2\pi)^3} \frac{e^{-iE|t|}e^{-i\vec{p}\cdot\vec{x}}}{2E}. \tag{62}$$

We now turn our attention to the more challenging integral over d^3p. Going to spherical coordinates, we note that $d^3p = p^2 dp d\Omega_2$.

$$\int\int \frac{p^2 dp}{(2\pi)^3} \frac{e^{-iE|t|}e^{-ipr\cos(\theta)}}{2E} d\Omega_2. \tag{63}$$

In 3 spacial dimensions, we can perform the angular integral straightforwardly writing $d\Omega_2 = d\cos(\theta)d\phi$. After the trivial integral over ϕ, we perform the integral over θ, then recognize the resulting difference of exponential as a sin. The result is

$$\int_0^{\infty} \frac{p^2 dp}{(2\pi)^2} \frac{e^{-iE|t|}}{E} \frac{\sin(pr)}{pr}. \tag{64}$$

We consult with Gradshtein and Ryzhik 3.194.9 and see that

$$\int_0^{\infty} \frac{xe^{-\beta\sqrt{\gamma^2+x^2}}}{4\pi^2\sqrt{\gamma^2+x^2}} \sin(bx)dx = \frac{\gamma b}{\sqrt{\beta^2+b^2}} K_1\left(\gamma\sqrt{\beta^2+b^2}\right). \tag{65}$$

Thus with $E = \sqrt{p^2+m^2}$, the propagator becomes

$$\frac{m}{4\pi^2\sqrt{r^2-t^2}} K_1\left(m\sqrt{r^2-t^2}\right), \tag{66}$$

which I rewrite in a Lorentzian form as

$$\frac{m}{4\pi^2\sqrt{-x^2}} K_1\left(m\sqrt{-x^2}\right). \tag{67}$$

To be thorough, we can separate the cases where x is spacelike and timelike. Using the relationship between the modified Bessel function and the Hankel function

$$K_n(x) = \frac{1}{2}\pi i^{n+1} H_n^{(1)}(ix), \tag{68}$$

we can *finally* write our full Feynman propagator in position space

$$D_F(x) = \theta(x^2)\frac{im}{8\pi\sqrt{x^2}} H_1^{(2)}(m\sqrt{x^2}) + \theta(-x^2)\frac{m}{4\pi^2(-x^2)^{1/2}} K_1(m(-x^2)^{1/2}). \tag{69}$$

(b)

To evaluate what happens near the light cone, $x^2 = 0$, we look at (64) in the massless limit

$$\frac{1}{r}\int_0^\infty \frac{dp}{(2\pi)^2} e^{-ip|t|}\sin(pr). \tag{70}$$

we again turn to Gradshtein and Ryzhik, this time 3.893.1

$$\int_0^\infty e^{-px}\sin(qx+\lambda)dx = \frac{1}{p^2+q^2}(q\cos\lambda + p\sin\lambda). \tag{71}$$

Our propagator becomes

$$\frac{1}{4\pi^2(r^2-t^2)}, \tag{72}$$

which for a null particle ($x^2 = t^2 - r^2 = 0$) is singular. Thus we can write it in the form

$$-\frac{1}{4\pi^2}\delta(x^2). \tag{73}$$

For completeness, we can now write the full Feynman propagator in position space as

$$D_F(x) = \theta(x^2)\frac{im}{8\pi\sqrt{x^2}}H_1^{(2)}(m\sqrt{x^2}) + \theta(-x^2)\frac{m}{4\pi^2(-x^2)^{1/2}}K_1(m(-x^2)^{1/2}) - \frac{1}{4\pi^2}\delta(x^2). \tag{74}$$

(c)

The Källen - Lehmann representation of the exact 2-point correlation function is

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty dm^2\rho(m^2)iD_F(x-y;m^2), \tag{75}$$

where $\rho(m^2) = \sum\delta(m^2 - m_\alpha^2)|\langle\Omega|\phi(0)|\alpha\rangle|^2$.

Near the light cone, we have

$$\int_0^\infty dm^2\sum_\alpha\delta(m^2-m_\alpha^2)|\langle\Omega|\phi(0)|\alpha\rangle|^2(-i)\frac{1}{4\pi^2}\delta(x^2) = -\frac{i}{4\pi^2}\sum_\alpha|\langle\Omega|\phi(0)|\alpha\rangle|^2\delta(x^2) \tag{76}$$

The sum over probabilities of the states simply gives 1, which means that near the light cone, the full 2-point correlation function of an interacting theory is $-i$ times the 2-point correlation function of the free theory!