Homework Set #5.

Due Date: Wednesday February 20, 2019

1. A non-relativistic beam of particles each with mass, m, and energy, E, which you can treat as a plane wave, is incident on a potential barrier of the form
$$U(x) = U_0 \quad (0 \leq x \leq L)$$
and
$$U(x) = 0 \quad (x < 0, x > L).$$
The incident beam energy is adjusted so that the special condition $E = U_0$ is obtained. Note that $U_0 > 0$.

   (a) Find the solutions to the Schrödinger equation in all three regions ($x < 0$, $0 \leq x \leq L$, and $x > L$).
   (b) In terms of the given quantities and any physical constants you need, what fraction of the beam is transmitted to the other side of the barrier?
   (c) What is your answer to part (b) in the limit $L \to \infty$?

2. Consider a particle of mass m in a one-dimensional infinite square well with a small delta-function barrier added in the middle. Take the width of the well to be $2a$, with the well centered at $x = 0$, and the amplitude of the delta function barrier to be $\alpha = \frac{\hbar^2}{10ma}$. Answer the first two parts before doing any calculations.

   A) Sketch the wavefunction of the ground state.

   B) Based on your sketch, should its energy be higher or lower than that of the ground-state energy of the same particle in a plain infinite square well of the same width? Explain.

   C) Set an upper limit on the energy of the ground state with the variational principle, using the wavefunction of the ground state of the plain well. By what percentage is this limit higher than the energy of the ground state of the plain well?

   D) Solve for the energy of the ground state exactly. What is the percent change relative to the plain well in this case?

   E) It might have seemed odd at first to call the delta function amplitude "small" when there is no scale to compare it to – the potential is zero or infinite everywhere. In retrospect, having solved the problem, what does "small" mean?

   Part D ends up requiring the solution to a very simple transcendental equation; one of these should prove useful:

$$\cot x = -5x \to x = 3.077 \qquad \cot x = 5x \to x = 3.204$$

$$\cot x = -10x \to x = 3.109 \qquad \cot x = 10x \to x = 3.173$$

$$\tan x = -5x \to x = 1.689 \qquad \tan x = 5x \to x = 1.432$$

$$\tan x = -10x \to x = 1.632 \qquad \tan x = 10x \to x = 1.504$$

3. The one-dimensional quantum mechanical potential energy of a particle of mass m is given by

$$V(x) = V_0\ \delta(x), \qquad -a < x < \infty,$$

$$V(x) = \infty, \qquad x < -a,$$

with a, V_0 positive, real constants.

At time t=0, the wave function of the particle is completely confined to the region $-a < x < 0$.

(a) Write down the normalized lowest-energy wave function of the particle at time $t = 0$.

(b) Give the boundary conditions which the energy eigenfunctions

$$\psi_k(x) = \psi_k^{\mathrm{I}}(x) \quad \text{and} \quad \psi_k(x) = \psi_k^{\mathrm{II}}(x)$$

must satisfy, where region I is $-a < x < 0$ and region II $x \geq 0$.

(c) Find the (real) solutions for the energy eigenfunctions in the two regions (up to an overall constant) which satisfy the boundary conditions.

(d) The $t = 0$ wave function can be expressed as an integral over energy eigenfunctions:

$$\psi(x) = \int_{-\infty}^{+\infty} f(k)\ \psi_k(x)\ dk.$$

Show how $f(k)$ can be determined from the solutions $\psi_k(x)$.

(e) Give an expression for the time development of the wave function in terms of $f(k)$. What values of k are expected to govern the time behavior at large times?

4. Find the wave function for a "free" particle of energy E moving in one dimension in a constant imaginary potential $-iV$, where $V \ll E$.

Calculate the probability current and show that an imaginary potential represents absorption of particles. Find an expression for the absorption coefficient in terms of V.

Note: problems (1-3) have been submitted or assigned as part of the UCSC Physics written qualifying exam for Quantum Mechanics.