Homework Set #6.

Due Date: Wednesday February 27, 2019

1. Define the partition function as

$$Z = \int d^3x' \; K(\vec{x'}, t; \vec{x'}, 0)|_{\beta = it/\hbar}.$$

Show that the ground-state energy is obtained by taking

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta}, \quad (\beta \to \infty).$$

Illustrate this for a particle in a one-dimensional box.

2. Consider the 2×2 matrix defined by

$$U = \frac{a_0 + i\vec{\sigma}\cdot\vec{a}}{a_0 - i\vec{\sigma}\cdot\vec{a}},$$

where $\vec{\sigma}$ are the usual three Pauli matrices, a_0 is a real number and $\vec{a}$ is a three-dimensional vector with real components.

   a. Prove that U is unitary and unimodular (det $= +1$).

   b. In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2 and a_3.

3. The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z direction can be written as

$$H = A\vec{S}^{(e^-)}\cdot\vec{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right).$$

Suppose the spin function of the system is given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$.

   a. Is this an eigenfunction of H in the limit $A \to 0$, $eB/mc \neq 0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of H?

   b. Same question, when $eB/mc \to 0$, $A \neq 0$.